

Summing Planar Diagrams by an Integrable Bootstrap II

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We continue our investigation of correlation functions of the large- N (planar) limit of the $(1+1)$ -dimensional principal chiral sigma model, whose bare field $U(x)$ lies in the fundamental matrix representation of $SU(N)$. We find all the form factors of the renormalized field $\Phi(x)$. An exact formula for Wightman and time-ordered correlation functions is found.

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I. INTRODUCTION

In Reference [1] (henceforth referred to as **I**), we found some form factors of the renormalized field Φ , of the $(1+1)$ -dimensional principal chiral model in 't Hooft's planar limit [2]. In particular, we obtained the one- and three-excitation form factors (the two-excitation form factor is zero). These form factors yield expressions for the correlation functions of the renormalized field, for large separations. In this paper we extend our results to all form factors of Φ . We thereby obtain exact expressions for correlation functions. Thus the planar diagrams are *completely* resummed.

Our technique is a combination of the form-factor axioms [3] and the $1/N$ -expansion of the exact S matrix [4], [5], [6]. A related development is the determination of the $1/N$ -expansion of the two- and four-excitation form factors of current operators, by A. Cortés Cubero [7].

We do not assume the reader is well-versed in form-factor lore, but take for granted acquaintance with integrability and two-dimensional S -matrix theory.

The bare field of the principal chiral model is a matrix $U(x) \in SU(N)$, $N \geq 2$, where x^0 and x^1 are the time and space coordinates, respectively. The action is

$$S = \frac{N}{2g_0^2} \int d^2x \, \eta^{\mu\nu} \text{Tr} \partial_\mu U(x)^\dagger \partial_\nu U(x), \quad (\text{I.1})$$

where $\mu, \nu = 0, 1$, $\eta^{00} = 1$, $\eta^{11} = -1$, $\eta^{01} = \eta^{10} = 0$, where g_0 is the coupling. This action is invariant under the global transformation $U(x) \rightarrow V_L U(x) V_R$, for two constant matrices $V_L, V_R \in SU(N)$. This field theory is asymptotically free, and we assume the existence of a mass gap m . The renormalized field operator $\Phi(x)$ is an average of $U(x)$ over a region of size b , where $\Lambda^{-1} < b \ll m^{-1}$, where Λ is an ultraviolet cutoff and m is the mass of the fundamental excitation.

Though $\Phi(x)$ is a complex $N \times N$ matrix, which is not directly proportional to the unitary matrix $U(x)$, we have the equivalence

$$\Phi(x) \sim Z(g_0, \Lambda)^{-1/2} U(x),$$

in the sense that

$$\frac{1}{N} \langle 0 | \text{Tr} \Phi(x) \Phi(0)^\dagger | 0 \rangle = Z(g_0, \Lambda)^{-1} \frac{1}{N} \langle 0 | \text{Tr} U(x) U(0)^\dagger | 0 \rangle. \quad (\text{I.2})$$

The renormalization factor $Z(g_0(\Lambda), \Lambda)$ goes to zero as $\Lambda \rightarrow \infty$ and the coupling $g_0(\Lambda)$ runs so that the mass gap $m(g_0(\Lambda), \Lambda)$ is independent of Λ .

The form factors may be combined into an expression for vacuum expectation values of products of $\Phi(x)$ and $\Phi(x)^\dagger$. We will use them to find an expression for the Wightman correlation function

$$\mathcal{W}(x) = \frac{1}{N} \langle 0 | \text{Tr} \Phi(0) \Phi(x)^\dagger | 0 \rangle. \quad (\text{I.3})$$

There are other integrable models for which Wightman functions have been found with the form-factor bootstrap. These include the sinh-Gordon model [8], the scaling limit of the two-dimensional Ising model [9] (for which other methods yield the same results [10]), the Z_N or clock model (a generalization of the Ising model to N states) [11], [12] and affine-Toda models [12].

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Form factors of $O(4) \simeq SU(2) \times SU(2)$ sigma models were investigated in References [13]. Form factors of the $SU(N)$ -invariant chiral Gross-Neveu model have been found in Reference [14] for arbitrary N .

Correlation functions in lower-dimensional models are not of interest only for their own sake. They have applications in situations where integrability is broken by interactions. What motivated our investigations was such an application of the $SU(N) \times SU(N)$ principal chiral model to $SU(N)$ gauge theories in $2 + 1$ dimensions [15].

In the next section, we present the $1/N$ -expansion of the S matrix, summarize the results of **I** and review (briefly) the Smirnov axioms. In Section 3, we find all the form factors as $N \rightarrow \infty$. All of these have simple poles in some relative rapidities. In Section 4, we apply our result to the large- N Wightman function. In Section 5, we find the time-ordered correlation function can be expressed as a sum of Feynman diagrams with massive propagators and spherical topology. We summarize our results and present a few open problems in the last section.

II. THE $1/N$ -EXPANSION AND FORM-FACTOR AXIOMS

In this section we briefly review the basic features of the principal chiral model in the planar limit, and summarize the results of **I**.

The S matrix of the elementary excitations of the principal chiral model [4], [5], [6] is written in terms of the incoming rapidities θ_1 and θ_2 (here $(p_j)_0 = m \cosh \theta_j$, $(p_j)_1 = m \sinh \theta_j$, relates the momentum vector p_j and rapidity θ_j), outgoing rapidities θ'_1 and θ'_2 and rapidity difference $\theta = |\theta_{12}| = |\theta_1 - \theta_2|$. The two-particle S matrix is

$$S_{PP}(\theta) = \frac{\sin(\theta/2 - \pi i/N)}{\sin(\theta/2 + \pi i/N)} S_{\text{CGN}}(\theta)_L \otimes S_{\text{CGN}}(\theta)_R, \quad (\text{II.1})$$

where $S_{\text{CGN}}(\theta)_{L,R}$, for either the subscript L (left) or R (right), is the S matrix of two elementary excitations, each of which is a vector N -plet, of the chiral Gross-Neveu model [16]:

$$S_{\text{CGN}}(\theta) = \frac{\Gamma(i\theta/2\pi + 1)\Gamma(-i\theta/2\pi - 1/N)}{\Gamma(i\theta/2\pi + 1 - 1/N)\Gamma(-i\theta/2\pi)} \left(\mathbb{1} - \frac{2\pi i}{N\theta} P \right),$$

and P interchanges the colors of the two chiral-Gross-Neveu N -plet excitations. The S matrix matrix of more than two excitations is built out of the two-particle S matrix using the factorization property, *i.e.* the Yang-Baxter equation.

The $1/N$ -expansion of the particle-particle S matrix (II.1) is [5]

$$S_{PP}(\theta)_{a_1 b_1; a_2 b_2}^{c_2 d_2; c_1 d_1} = [1 + O(1/N^2)] \left[\delta_{a_2}^{c_2} \delta_{b_2}^{d_2} \delta_{a_1}^{c_1} \delta_{b_1}^{d_1} - \frac{2\pi i}{N\theta} \left(\delta_{a_1}^{c_2} \delta_{b_2}^{d_2} \delta_{a_2}^{c_1} \delta_{b_1}^{d_1} + \delta_{a_2}^{c_2} \delta_{b_1}^{d_2} \delta_{a_1}^{c_1} \delta_{b_2}^{d_1} \right) - \frac{4\pi^2}{N^2 \theta^2} \delta_{a_1}^{c_2} \delta_{b_1}^{d_2} \delta_{a_2}^{c_1} \delta_{b_2}^{d_1} \right]. \quad (\text{II.2})$$

The scattering matrix of one particle and one antiparticle $S_{PA}(\theta)$ is obtained by crossing (II.2) from the s -channel to the t -channel:

$$S_{PA}(\theta)_{a_1 b_1; b_2 a_2}^{d_2 c_2; c_1 d_1} = [1 + O(1/N^2)] \times \left[\delta_{b_2}^{d_2} \delta_{a_2}^{c_2} \delta_{a_1}^{c_1} \delta_{b_1}^{d_1} - \frac{2\pi i}{N\hat{\theta}} \left(\delta_{a_1 a_2} \delta^{c_1 c_2} \delta_{b_2}^{d_2} \delta_{b_1}^{d_1} + \delta_{a_2}^{c_2} \delta_{a_1}^{c_1} \delta_{b_1 b_2} \delta^{d_1 d_2} \right) - \frac{4\pi^2}{N^2 \hat{\theta}^2} \delta_{a_1 a_2} \delta^{c_1 c_2} \delta_{b_1 b_2} \delta^{d_1 d_2} \right], \quad (\text{II.3})$$

where $\hat{\theta} = \pi i - \theta$ is the crossed rapidity difference. As in **I**, we define the generalized S matrix by replacing $\theta = |\theta_{12}|$ with $\theta = \theta_{12}$.

The large- N limit we consider is the standard 't Hooft limit. We assume the mass gap is fixed, as N is taken to infinity.

There are bound states of the elementary particles, corresponding to poles of the S matrix. As we mentioned in **I**, only one bound state plays a role in the correlation functions of Φ . This bound state, namely the antiparticle, consists of $N - 1$ fundamental particles.

The first form factor (discussed in **I**) is the simple normalization condition

$$\langle 0 | \Phi(0)_{b_0 a_0} | P, \theta, a_1, b_1 \rangle = N^{-1/2} \delta_{a_0 a_1} \delta_{b_0 b_1}, \quad (\text{II.4})$$

where the ket on the right is a one-particle state, with rapidity θ and left and right colors a_1 and b_1 , respectively.

The S matrix can be determined, assuming unitarity, factorization (the Yang-Baxter relation) and maximal analyticity. The excitations which survive in the $N \rightarrow \infty$ limit have two color indices from 1 to N . One can view these excitations as a bound pair of two quarks of different color sectors (or alternatively as a quark in one color sector and an antiquark in the other).

The Zamolodchikov algebra is spanned by particle-creation operators $\mathfrak{A}_P^\dagger(\theta)_{ab}$ and antiparticle-creation operators $\mathfrak{A}_A^\dagger(\theta)_{ba}$. These operators satisfy what is essentially a non-Abelian parastatistics relation:

$$\begin{aligned}\mathfrak{A}_P^\dagger(\theta_1)_{a_1 b_1} \mathfrak{A}_P^\dagger(\theta_2)_{a_2 b_2} &= S_{PP}(\theta_{12})_{a_1 b_1; a_2 b_2}^{c_2 d_2; c_1 d_1} \mathfrak{A}_P^\dagger(\theta_2)_{c_2 d_2} \mathfrak{A}_P^\dagger(\theta_1)_{c_1 d_1} \\ \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} &= S_{AA}(\theta_{12})_{b_1 a_1; b_2 a_2}^{d_2 c_2; d_1 c_1} \mathfrak{A}_A^\dagger(\theta_2)_{d_2 c_2} \mathfrak{A}_A^\dagger(\theta_1)_{d_1 c_1} \\ \mathfrak{A}_P^\dagger(\theta_1)_{a_1 b_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} &= S_{PA}(\theta_{12})_{a_1 b_1; b_2 a_2}^{d_2 c_2; c_1 d_1} \mathfrak{A}_A^\dagger(\theta_2)_{d_2 c_2} \mathfrak{A}_P^\dagger(\theta_1)_{c_1 d_1}.\end{aligned}\quad (\text{II.5})$$

Consistency of this algebra implies the Yang-Baxter equation.

An in-state is defined as a product of creation operators in the order of increasing rapidity, from right to left, acting on the vacuum, *e.g.*

$$|P, \theta_1, a_1, b_1; A, \theta_2, b_2, a_2, \dots\rangle_{\text{in}} = \mathfrak{A}_P^\dagger(\theta_1)_{a_1 b_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} \dots |0\rangle, \quad \text{where } \theta_1 > \theta_2 > \dots \quad (\text{II.6})$$

Similarly, an out-state is a product of creation operators in the order of decreasing rapidity, from right to left, acting on the vacuum.

The S matrix becomes the unit operator as $N \rightarrow \infty$. Hence the basic dynamical field is

$$M(x) = \int \frac{d\theta}{2\pi} \left[\mathfrak{A}_P(\theta) e^{imx^0 \cosh \theta - imx^1 \sinh \theta} + \mathfrak{A}_A^\dagger(\theta) e^{-imx^0 \cosh \theta + imx^1 \sinh \theta} \right], \quad (\text{II.7})$$

where \mathfrak{A}_P is the destruction operator of a particle (which is the adjoint of the operator \mathfrak{A}_P^\dagger). We can think of this $N \times N$ matrix-field operator as the master field, since its response to an external source is the same as that of a classical field (we can similarly regard the free field appearing in the large- N limit of an $O(N)$ -symmetric model as the master field). In **I**, we pointed out that the form factors express the renormalized field $\Phi(x)$ in terms of the field $M(x)$.

The form factors are matrix elements between the vacuum and multi-particle in-states of the field operator Φ . The action of the global-symmetry transformation on Φ and the creation operators is

$$\Phi(x) \rightarrow V_L \Phi(x) V_R, \quad \mathfrak{A}_P^\dagger(\theta) \rightarrow V_R^\dagger \mathfrak{A}_P^\dagger(\theta) V_L^\dagger, \quad \mathfrak{A}_A^\dagger(\theta) \rightarrow V_L \mathfrak{A}_A^\dagger(\theta) V_R. \quad (\text{II.8})$$

Thus we expect that, for large N , the condition

$$\langle 0 | \Phi(0) | \Psi \rangle \neq 0,$$

on an in-state $|\Psi\rangle$, which is an eigenstate of particle number, holds only if $|\Psi\rangle$ contains M particles and $M - 1$ antiparticles, for some $M = 1, 2, \dots$. In **I** we found these matrix elements for $M = 1$ (equation (II.4) above) and $M = 2$:

$$\begin{aligned}\langle 0 | \Phi(0)_{b_0 a_0} | A, \theta_1, b_1, a_1; P, \theta_2, a_2, b_2; P, \theta_3, a_3, b_3 \rangle_{\text{in}} &= \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_P^\dagger(\theta_2)_{a_2 b_2} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} | 0 \rangle \\ &= \frac{1}{N^{3/2}} F_1(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_2} \delta_{b_0 b_3} \delta_{b_1 b_2} \delta_{a_1 a_3} + \frac{1}{N^{3/2}} F_2(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_3} \delta_{b_0 b_2} \delta_{a_1 a_2} \delta_{b_1 b_3} \\ &\quad + \frac{1}{N^{3/2}} F_3(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_2} \delta_{b_0 b_2} \delta_{a_1 a_3} \delta_{b_1 b_3} + \frac{1}{N^{3/2}} F_4(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_3} \delta_{b_0 b_3} \delta_{b_1 b_2} \delta_{a_1 a_2},\end{aligned}\quad (\text{II.9})$$

for $\theta_1 > \theta_2 > \theta_3$,

$$\begin{aligned}\langle 0 | \Phi(0)_{b_0 a_0} | P, \theta_1, a_1, b_1; A, \theta_2, b_2, a_2; P, \theta_3, a_3, b_3 \rangle_{\text{in}} &= \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_P^\dagger(\theta_2)_{a_2 b_2} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} | 0 \rangle \\ &= \frac{1}{N^{3/2}} \tilde{F}_1(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_2} \delta_{b_0 b_3} \delta_{b_1 b_2} \delta_{a_1 a_3} + \frac{1}{N^{3/2}} \tilde{F}_2(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_3} \delta_{b_0 b_2} \delta_{a_1 a_2} \delta_{b_1 b_3} \\ &\quad + \frac{1}{N^{3/2}} \tilde{F}_3(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_2} \delta_{b_0 b_2} \delta_{a_1 a_3} \delta_{b_1 b_3} + \frac{1}{N^{3/2}} \tilde{F}_4(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_3} \delta_{b_0 b_3} \delta_{b_1 b_2} \delta_{a_2 a_1},\end{aligned}\quad (\text{II.10})$$

for $\theta_2 > \theta_1 > \theta_3$, and

$$\begin{aligned}\langle 0 | \Phi(0)_{b_0 a_0} | P, \theta_1, a_1, b_1; P, \theta_2, a_2, b_2; A, \theta_3, b_3, a_3 \rangle_{\text{in}} &= \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_P^\dagger(\theta_2)_{a_2 b_2} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} | 0 \rangle \\ &= \frac{1}{N^{3/2}} \tilde{\tilde{F}}_1(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_2} \delta_{b_0 b_3} \delta_{b_1 b_2} \delta_{a_1 a_3} + \frac{1}{N^{3/2}} \tilde{\tilde{F}}_2(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_3} \delta_{b_0 b_2} \delta_{a_1 a_2} \delta_{b_1 b_3} \\ &\quad + \frac{1}{N^{3/2}} \tilde{\tilde{F}}_3(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_2} \delta_{b_0 b_2} \delta_{a_1 a_3} \delta_{b_1 b_3} + \frac{1}{N^{3/2}} \tilde{\tilde{F}}_4(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_3} \delta_{b_0 b_3} \delta_{b_1 b_2} \delta_{a_2 a_1},\end{aligned}\quad (\text{II.11})$$

for $\theta_3 > \theta_1 > \theta_2$. where

$$\begin{aligned} F_1(\theta_1, \theta_2, \theta_3) &= -\frac{4\pi}{(\theta_{12} + \pi i)(\theta_{13} + \pi i)} + O(1/N), \quad F_2(\theta_1, \theta_2, \theta_3) = -\frac{4\pi}{(\theta_{12} + \pi i)(\theta_{13} + \pi i)} + O(1/N), \\ F_3(\theta_1, \theta_2, \theta_3) &= O(1/N), \quad F_4(\theta_1, \theta_2, \theta_3) = O(1/N), \end{aligned} \quad (\text{II.12})$$

and, to order $1/N^0$, the functions with one or two tildes are the same as those in (II.12) except for phases. We should mention that the vanishing of F_3 and F_4 is essential. If these quantities were not zero, the double poles would lead to diverging, unphysical S matrix elements through the LSZ reduction formula.

Here is a quick (but incomplete) summary of Smirnov's form-factor axioms [3] for arbitrary particle states:

Scattering Axiom (Watson's theorem). From the Zamolodchikov algebra (II.5),

$$\begin{aligned} \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_{I_1}^\dagger(\theta_1)_{C_1} \cdots \mathfrak{A}_{I_j}^\dagger(\theta_j)_{C_j} \mathfrak{A}_{I_{j+1}}^\dagger(\theta_{j+1})_{C_{j+1}} \cdots \mathfrak{A}_{I_M}^\dagger(\theta_M)_{C_M} | 0 \rangle \\ = S_{I_j I_{j+1}}(\theta_j, \theta_{j+1})_{C_j C_{j+1}}^{C'_j C'_{j+1}} \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_{I_1}^\dagger(\theta_1)_{C_1} \cdots \mathfrak{A}_{I'_j}^\dagger(\theta_{j+1})_{C'_{j+1}} \mathfrak{A}_{I'_j}^\dagger(\theta_j)_{C'_j} \cdots \mathfrak{A}_{I_M}^\dagger(\theta_M)_{C_M} | 0 \rangle, \end{aligned} \quad (\text{II.13})$$

where $I_k, k = 1, \dots, M$ is P or A (particle or antiparticle) and C_k denotes a pair of indices (which may be written $a_k b_k$, for $C_k = P$ and $b_k a_k$, for $C_k = A$) and similarly for the primed indices.

Periodicity Axiom.

$$\langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_{I_1}^\dagger(\theta_1)_{C_1} \mathfrak{A}_{I_2}^\dagger(\theta_2)_{C_2} \cdots \mathfrak{A}_{I_n}^\dagger(\theta_n)_{C_n} | 0 \rangle = \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_{I_n}^\dagger(\theta_n - 2\pi i)_{C_n} \mathfrak{A}_{I_1}^\dagger(\theta_1)_{C_1} \cdots \mathfrak{A}_{I_{n-1}}^\dagger(\theta_{n-1})_{C_{n-1}} | 0 \rangle. \quad (\text{II.14})$$

Annihilation-Pole Axiom. This is a recursive relation, which fixes the residues of the poles of the form factors. This axiom and the previous one are special cases of a generalized crossing formula obtained in Reference [17].

$$\begin{aligned} \text{Res}_{|\theta_{1n} = -\pi i} \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_{I_1}^\dagger(\theta_1)_{C_1} \mathfrak{A}_{I_2}^\dagger(\theta_2)_{C_2} \cdots \mathfrak{A}_{I_n}^\dagger(\theta_n)_{C_n} | 0 \rangle \\ = -2i \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_{I_2}^\dagger(\theta_2)_{C'_2} \mathfrak{A}_{I_3}^\dagger(\theta_2)_{C'_3} \cdots \mathfrak{A}_{I_{n-1}}^\dagger(\theta_{n-1})_{C'_{n-1}} | 0 \rangle \\ \times \left[S_{I_1 I_2}(\theta_{12})_{C_1 C_2}^{C'_1 D_1} S_{I_1 I_3}(\theta_{13})_{D_1 C_3}^{C'_1 D_2} \cdots S_{I_1 I_{n-1}}(\theta_{1, n-1})_{D_{n-2} C_{n-1}}^{C'_1 D_{n-1}} - \delta_{C_1}^{C'_1} \delta_{C_2}^{C'_2} \delta_{C_3}^{C'_3} \cdots \delta_{C_{n-1}}^{C'_{n-1}} \right], \end{aligned} \quad (\text{II.15})$$

If we assume the Lehmann-Symanzik-Zimmermann (LSZ) formula for the connected part of the S matrix with $n - 2$ external lines, then (II.15) implies a similar LSZ formula with n external lines. See I for some discussion of the relation between this axiom and the reduction formula, in the context of the large- N limit of the principal chiral model.

Lorentz-Invariance Axiom. For the scalar operator Φ , this takes the form

$$\langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_{I_1}^\dagger(\theta_1 + \Delta\theta)_{C_1} \cdots \mathfrak{A}_{I_M}^\dagger(\theta_M + \Delta\theta)_{C_M} | 0 \rangle = \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_{I_1}^\dagger(\theta_1)_{C_1} \cdots \mathfrak{A}_{I_M}^\dagger(\theta_M)_{C_M} | 0 \rangle, \quad (\text{II.16})$$

for an arbitrary boost $\Delta\theta$.

Bound-State Axiom. This axiom says that there are poles on the imaginary axis of rapidity differences θ_{jk} , due to bound states. we will not discuss it further, because there are no bound states in the 't Hooft limit.

Minimality Axiom. In general, form factors are holomorphic, except possibly for bound-state poles, for rapidity differences θ_{jk} in the complex strip $0 < \text{Im } \theta_{jk} < 2\pi$. The minimality axiom states that form factors have as much analyticity as is consistent with the other axioms.

Some discussion of the meaning and use of these axioms in the context of the $N \rightarrow \infty$ limit can be found in I.

III. FORM FACTORS FOR GENERAL IN-STATES

The general matrix element of $\Phi(0)$ between the vacuum and an $(M - 1)$ -antiparticle, M -particle state has many terms. By comparing it to the S -matrix element describing the scattering of M -particles, we can determine the most significant part of the form factors for large N . This part is proportional to $N^{-M+1/2}$. We denote left and right permutations (in the permutation group S_M) by σ and τ , respectively. We use the convention that σ and τ take the set of numbers $0, 1, 2, \dots, M - 1$ to $\sigma(0), \sigma(1), \dots, \sigma(M - 1)$ and $\tau(0), \tau(1), \dots, \tau(M - 1)$, respectively. The most general form factor of the renormalized field is

$$\langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_{I_1}^\dagger(\theta_1)_{C_1} \mathfrak{A}_{I_2}^\dagger(\theta_2)_{C_2} \cdots \mathfrak{A}_{I_{2M-1}}^\dagger(\theta_{2M-1})_{C_{2M-1}} | 0 \rangle = \frac{\sqrt{N}}{N^M} \sum_{\sigma, \tau \in S_M} F_{\sigma\tau}(\theta_1, \theta_2, \dots, \theta_{2M-1}) \prod_{j=0}^{M-1} \delta_{a_j a_{\sigma(j)+M}} \delta_{b_j b_{\tau(j)+M}}. \quad (\text{III.1})$$

The order $(1/N)^0$ parts of the coefficients of the tensors

$$N^{-M+1/2} \prod_{j=0}^{M-1} \delta_{a_j a_{\sigma(j)+M}} \delta_{b_j b_{\tau(j)+M}}, \quad (\text{III.2})$$

that is $F_{\sigma\tau}^0(\theta_1, \theta_2, \dots, \theta_{2M-1})$, are the same, no matter the order of the creation operators on the left-hand side of (III.1), except for a phase, as we explain below.

The function $F_{\sigma\tau}$ can be expanded in powers of $1/N$, *i.e.*

$$F_{\sigma\tau}(\theta_1, \theta_2, \dots, \theta_{2M-1}) = F_{\sigma\tau}^0(\theta_1, \theta_2, \dots, \theta_{2M-1}) + \frac{1}{N} F_{\sigma\tau}^1(\theta_1, \theta_2, \dots, \theta_{2M-1}) + \dots, \quad (\text{III.3})$$

We only consider only the leading term on the right-hand side of (III.3) here.

Suppose we interchange two adjacent creation operators in the left-hand side of (III.1). The scattering axiom (II.13) implies that as $N \rightarrow \infty$:

1. If both creation operators create an antiparticle or both operators create a particle, the result is the interchange of the rapidities of these two operators, in the function $F_{\sigma\tau}^0$.
2. If one operator creates an antiparticle with rapidity θ_j and colors a_j, b_j and the other operator creates a particle with rapidity θ_k and colors a_k, b_k , and $\sigma(j) + M \neq k, \tau(j) + M \neq k$, there is no effect on the function $F_{\sigma\tau}^0$.
3. If one operator creates an antiparticle with rapidity θ_j and colors a_j, b_j and the other operator creates a particle with rapidity θ_k and colors a_k, b_k , and $\sigma(j) + M = k, \tau(j) + M \neq k$, then $F_{\sigma\tau}^0$ is multiplied by the phase $\frac{\theta_{jk} + \pi i}{\theta_{jk} - \pi i}$.
4. If one operator creates an antiparticle with rapidity θ_j and colors a_j, b_j and the other operator creates a particle with rapidity θ_k and colors a_k, b_k , and $\sigma(j) + M \neq k, \tau(j) + M = k$, then $F_{\sigma\tau}^0$ is multiplied by the phase $\frac{\theta_{jk} + \pi i}{\theta_{jk} - \pi i}$.
5. If one operator creates an antiparticle with rapidity θ_j and colors a_j, b_j and the other operator creates a particle with rapidity θ_k and colors a_k, b_k , and $\sigma(j) + M = k, \tau(j) + M = k$, then $F_{\sigma\tau}^0$ is multiplied by the phase $\left(\frac{\theta_{jk} + \pi i}{\theta_{jk} - \pi i}\right)^2$.

Statements 1. through 5. above are straightforward generalizations of the $M = 2$ case, discussed in I. The interchange of two creation operators has no effect at leading order in $1/N$, unless indices are contracted to make a factor of N . This factor of N compensates for the terms of order $1/N$ in the S matrix. If no indices are contracted, the only part of the S matrix which contributes is unity. If the creation operators have an index in common, the phase $\frac{\theta_{jk} + \pi i}{\theta_{jk} - \pi i}$ appears, just as for the $M = 2$ case. If two indices are contracted, this phase is squared. This is why Watson's theorem is meaningful as $N \rightarrow \infty$, despite there being no scattering!

Consider the following:

$$\begin{aligned} & \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \cdots \mathfrak{A}_A^\dagger(\theta_{M-1})_{b_{M-1} a_{M-1}} \mathfrak{A}_P^\dagger(\theta_M)_{a_M b_M} \cdots \mathfrak{A}_P^\dagger(\theta_{2M-1})_{a_{2M-1} b_{2M-1}} | 0 \rangle \\ &= N^{-M+1/2} \sum_{\sigma, \tau \in S_M} F_{\sigma\tau}(\theta_1, \theta_2, \dots, \theta_{2M-1}) \prod_{j=0}^{M-1} \delta_{a_j a_{\sigma(j)+M}} \delta_{b_j b_{\tau(j)+M}}, \end{aligned} \quad (\text{III.4})$$

which is a special case of (III.1). We interchange the leftmost creation operator $\mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1}$ consecutively with all the other creation operators. In other words, we are “pushing” $\mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1}$ to the right, past all the other creation operators. The periodicity axiom for θ_1 , together with the conditions 1. through 5., implies that the $N \rightarrow \infty$ limit of the function $F_{\sigma\tau}$ in (III.4) has the following structure, as a function of θ_1

$$F_{\sigma\tau}^0(\theta_1, \theta_2, \dots, \theta_{2M-1}) \sim [\theta_1 - \theta_{\sigma(1)+M} + \pi i]^{-1} [\theta_1 - \theta_{\tau(1)+M} + \pi i]^{-1} h(\theta_1, \dots, \theta_{2M-1})$$

where $h(\theta_1, \dots, \theta_{2M-1})$ is some function which is analytic and periodic, with period $2\pi i$, in $\theta_1, \theta_{\sigma(1)+M}$ and $\theta_{\tau(1)+M}$.

Now suppose we start with the same expression, namely (III.4), and interchange $\mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1}$ with $\mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2}$. This has the effect on the function $F_{\sigma\tau}$ of interchanging the arguments θ_1 and θ_2 . Then we can apply a procedure similar to that of the last paragraph, pushing $\mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2}$ all the way to the right. We repeat this procedure for all the creation operators for antiparticles. We conclude that $F_{\sigma\tau}^0$ has the form

$$F_{\sigma\tau}^0(\theta_1, \theta_2, \dots, \theta_{2M-1}) = \frac{g_{\sigma\tau}(\theta_1, \dots, \theta_{2M-1})}{\prod_{j=1}^{M-1} [\theta_j - \theta_{\sigma(j)+M} + \pi i][\theta_j - \theta_{\tau(j)+M} + \pi i]}, \quad (\text{III.5})$$

where $g_{\sigma\tau}(\theta_1, \dots, \theta_{2M-1})$ is holomorphic and periodic in each θ_j , $j = 1, \dots, 2M-1$, with period $2\pi i$. In (III.5), there are possible poles occurring at $\theta_j - \theta_{\sigma(j)+M} = -\pi i$ and $\theta_j - \theta_{\tau(j)+M} = -\pi i$. The maximum number of such poles on the right-hand side of (III.5), including multiplicity, is $2M-2$. This is precisely the number needed to generate the connected part of the S matrix (from the reduction formula) to leading order in $1/N$.

A choice of (III.5), satisfying the annihilation-pole axiom and having as much analyticity as possible is

$$F_{\sigma\tau}^0(\theta_1, \theta_2, \dots, \theta_{2M-1}) = \frac{(-4\pi)^{M-1} K_{\sigma\tau}}{\prod_{j=1}^{M-1} [\theta_j - \theta_{\sigma(j)+M} + \pi i] [\theta_j - \theta_{\tau(j)+M} + \pi i]}, \quad (\text{III.6})$$

where

$$K_{\sigma\tau} = \begin{cases} 1, & \sigma(j) \neq \tau(j), \text{ for all } j \\ 0, & \text{otherwise} \end{cases}. \quad (\text{III.7})$$

Notice that the expression for $K_{\sigma\tau}$ insures the absence of double poles. We recover (II.12) for $M=1$. Thus $K_{\sigma\tau}$ is unity if and only if the permutation $\sigma\tau^{-1}$ has no fixed points, *i.e.* has the smallest possible fundamental character in S_M . The number of pairs σ and τ in S_M satisfying this condition is $(M-1)!M!$. Together, (III.4), (III.6) and (III.7) yield the form factors.

IV. WIGHTMAN FUNCTIONS

The Wightman function is obtained from the form factors using the completeness of in-states:

$$\mathcal{W}(x) = \frac{1}{N} \sum_{a_0, b_0} \sum_X \langle 0 | \Phi(0)_{b_0 a_0} | X \rangle_{\text{in}} \langle X | [\Phi(0)_{b_0 a_0}]^* | 0 \rangle e^{ip_X \cdot x} = \frac{1}{N} \sum_{a_0, b_0} \sum_X |\langle 0 | \Phi(0)_{b_0 a_0} | X \rangle_{\text{in}}|^2 e^{ip_X \cdot x},$$

where X denotes an arbitrary choice of particles, momenta and colors and where p_X is the momentum eigenvalue of the state $|X\rangle$. From the result of the last section,

$$\mathcal{W}(x) = \sum_{M=1}^{\infty} \frac{1}{(M-1)!} \frac{1}{M!} \int \left(\prod_{j=1}^{2M-1} \frac{d\theta_j}{4\pi} \right) \sum_{\sigma\tau} \frac{(4\pi)^{2M-2} K_{\sigma\tau} \exp(i x \cdot \sum_{j=1}^{2M-1} p_j)}{\prod_{j=1}^{M-1} |\theta_j - \theta_{\sigma(j)+M} + \pi i|^2 |\theta_j - \theta_{\tau(j)+M} + \pi i|^2} + O(1/N), \quad (\text{IV.1})$$

where $p_j = m(\cosh \theta_j, \sinh \theta_j)$. Notice that the leading term is of order $1/N^0$. We did the sum over all color indices on the right-hand side of (IV.1), using

$$\sum_{a_0, \dots, a_{2M-1}, b_0, \dots, b_{2M-1}} K_{\sigma\tau} K_{\omega\phi} \left[\prod_{j=0}^{M-1} \delta_{a_j a_{\sigma(j)+M}} \delta_{b_j b_{\tau(j)+M}} \right] \left[\prod_{j=0}^{M-1} \delta_{a_j a_{\omega(j)+M}} \delta_{b_j b_{\phi(j)+M}} \right] = N^{2M-2} K_{\sigma\tau} [\delta_{\sigma\omega} \delta_{\tau\phi} + O(1/N)], \quad (\text{IV.2})$$

for permutations $\sigma, \tau, \omega, \phi \in S_M$. The $M=2$ case of (IV.2) was discussed in I. This relation tells us that the sum over the product of two of the color tensors (III.2) will not contribute as $N \rightarrow \infty$, unless they are the same tensor.

We can further simplify Equation (IV.1). Each contribution from a pair of permutations σ, τ , satisfying $K_{\sigma\tau} = 1$ on the right-hand side of (IV.1) is the same, after integrating over the rapidities $\theta_1, \dots, \theta_{2M-1}$. We can therefore pick one pair of permutations and multiply by $M!(M-1)!$ (canceling a similar factor in the denominator). We choose the identity for σ and a cyclic permutation for τ :

$$\sigma(j) = j, \quad \tau(j) = j+1 \pmod{M}, \text{ for } j = 0, \dots, M-1.$$

The Wightman function is therefore

$$\begin{aligned} \mathcal{W}(x) = & \int \frac{d\theta}{4\pi} e^{ix \cdot p} + \frac{1}{4\pi} \sum_{M=2}^{\infty} \int d\theta_1 \cdots d\theta_{2M-1} e^{ix \cdot \sum_{j=1}^{2M-1} p_j} \frac{1}{(\theta_1 - \theta_M)^2 + \pi^2} \frac{1}{(\theta_M - \theta_2)^2 + \pi^2} \frac{1}{(\theta_2 - \theta_{M+1})^2 + \pi^2} \\ & \times \frac{1}{(\theta_{M+1} - \theta_3)^2 + \pi^2} \cdots \frac{1}{(\theta_{M-2} - \theta_{2M-1})^2 + \pi^2} \frac{1}{(\theta_{2M-1} - \theta_{M-1})^2 + \pi^2}. \end{aligned}$$

The first two terms of this series were presented in I. We relabel the indices on rapidities by

$$\theta_1 \rightarrow \theta_1, \theta_M \rightarrow \theta_2, \theta_2 \rightarrow \theta_3, \dots, \theta_{2M-1} \rightarrow \theta_{2M-2}, \theta_{M-1} \rightarrow \theta_{2M-1},$$

obtaining

$$\mathcal{W}(x) = \int \frac{d\theta}{4\pi} e^{im(x^- e^\theta + x^+ e^{-\theta})} + \frac{1}{4\pi} \sum_{l=1}^{\infty} \int d\theta_1 \cdots d\theta_{2l+1} \exp \left[i \sum_{j=1}^{2l+1} m(x^- e^{\theta_j} + x^+ e^{-\theta_j}) \right] \prod_{j=1}^{2l} \frac{1}{(\theta_j - \theta_{j+1})^2 + \pi^2}, \quad (\text{IV.3})$$

where $l = M - 1$ and $x^\pm = (x^0 \pm x^1)/2$ are light-cone coordinates.

The terms in the series (IV.3) are multiple integrals over the Poisson kernel for the upper half-plane. Suppose that $f(\theta)$ is a function of real θ , such that $|f(\theta)| \leq C$, for some real positive constant C . The Poisson kernel integrated over f is

$$Pf(\theta, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\theta' \frac{yf(\theta')}{(\theta - \theta')^2 + y^2}, \quad y \geq 0.$$

This function is harmonic everywhere in the upper half-plane, with the Dirichlet boundary condition $Pf(\theta, 0) = f(\theta)$. The terms in (IV.3) are repeated integrations of this type, with $y = \pi$. This guarantees that each term in the series is finite (though it does not guarantee convergence of the series). The subtlety in evaluating the terms in the series (IV.3) is that $\exp im(x^- e^{\theta_j} + x^+ e^{-\theta_j})$, an analytic function of complex θ_j , is not bounded in the upper half-plane as $|\theta_j| \rightarrow \infty$. In particular, as $\Re \theta_j \rightarrow \pm\infty$, it diverges for some choices of $\Im \theta_j$. The integral over the Poisson kernel, however, is bounded and harmonic, but not analytic.

V. TIME-ORDERED GREEN'S FUNCTIONS

If we time-order the fields of the correlation function of the last section, we replace the time coordinate x^0 by $|x^0|$. The Lorentz-invariant two-point Green's function:

$$G(x) = \frac{1}{N} \langle 0 | T \text{Tr} \Phi(0) \Phi(x) | 0 \rangle,$$

can be written as a sum of integrals over energy-momentum two-vectors, $p_1, p_2, \dots, p_{2l+1}$:

$$G(x) = \sum_{l=0}^{\infty} \frac{1}{4\pi} \int d^2 p_1 d^2 p_2 \cdots d^2 p_{2l+1} e^{-i(p_1 + p_2 + \cdots + p_{2l+1}) \cdot x} \left[\prod_{j=1}^{2l} \left|_{j \text{ even}} \frac{1}{(\cosh^{-1} \frac{p_j \cdot p_{j+1}}{m^2})^2 + \pi^2} \right] \right. \\ \times \frac{i}{\pi(p_1^2 - m^2 + i\epsilon)} \frac{i}{\pi(p_2^2 - m^2 + i\epsilon)} \cdots \frac{i}{\pi(p_{2l+1}^2 - m^2 + i\epsilon)} \left. \left[\prod_{j=1}^{2l} \left|_{j \text{ odd}} \frac{1}{(\cosh^{-1} \frac{p_j \cdot p_{j+1}}{m^2})^2 + \pi^2} \right] \right], \quad (\text{V.1})$$

where $p_j \cdot p_k = \eta^{\mu\nu} p_{j\mu} p_{k\nu}$ and $p_j^2 = p_j \cdot p_j$. Each finite term in this amplitude is given by a rainbow-type Feynman diagram, with two vertices and $2l$ loops. For $l = 3$, this diagram is



Each vertex is of order $2l + 1$ (in other words, is joined by $2l + 1$ propagators). The massive propagators in (V.1) are those of the $l + 1$ particles and l antiparticles joining the two vertices. Though such diagrams, for $l > 0$, are one-particle irreducible, (V.1) is the connected Green's function, not the one-particle-irreducible Green's function.

VI. DISCUSSION

In this paper, we extended the derivation of the one- and three-excitation form factors of the renormalized field in **I** to all the form factors of the this field. Using these form factors, we found expressions for correlation functions.

It is important to know the behavior of the two-point Wightman function at short distances. Its Fourier transform, as a function of momentum q , must be consistent with asymptotic freedom. In particular, this function should be $\sim \sqrt{\log|q^2|/q^2}$, for large q . We hope to check that this behavior follows from (IV.3). If this can be done, it seems feasible to find coefficients of operator-product expansions. For example, we expect that for small x ,

$$\Phi(0)\Phi(x)^\dagger \simeq \mathcal{W}(x)[\mathbb{1} + x^\mu U(0)\partial_\mu U(0)^\dagger] + \cdots = \mathcal{W}(x)[\mathbb{1} + ix^\mu j_\mu^L(0)] + \cdots. \quad (\text{VI.1})$$

where $j_\mu^L(x) = -iU(x)\partial_\mu U(x)^\dagger$ is the left-handed current. The normalization of the second term in (VI.1) should be consistent with the $SU(\infty)$ current algebra. The completeness of in-states makes it possible to check that the form factors (III.6), (III.7) are consistent with the form factors of current operators [7]. The latter form factors should be useful in the study of the large- N limit of $SU(N)$ gauge theories in $2 + 1$ dimensions, along the lines of References [15].

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